Discrete Element Idealization of an Incompressible Liquid for Vibration Analysis

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A new type of structural element for discrete element idealization of an incompressible liquid is introduced. With this liquid structural element and the standard structural elements, a missile and liquid propellant can be idealized as one composite structure consisting of many discrete elements. This idealization would be valid for all linear displacement modes of vibration. Some of the modes, the number depending on the number of liquid elements, are zero frequency modes. This reflects a true liquid characteristic. The new type of idealization is not used as the basis for analysis in the usual manner. Instead, the idealization leads to a new method for computing a liquid mass matrix with respect to the container coordinates. The resulting characteristic value problem of the vehicle containing liquid is the same size as for the empty vehicle. For illustration, the static and dynamic properties of a simple container and liquid structure are investigated.

Introduction

CURRENT methods of liquid propellant idealization for missile vibration studies are based on a liquid model which approximates fluid motion for only one particular mode, say for the axisymmetric mode.[1,2] Another idealization would be necessary for the bending mode. Naturally, the idealization is not very good for modes that do not correspond to the assumed mode.

The purpose of this paper is to introduce a new type of structural element for discrete element idealization of an incompressible liquid for vibration studies. With this liquid structural element and the standard structural elements, a missile and liquid propellant can be idealized as one composite structure consisting of many discrete elements. This idealization would be valid for all linear displacement modes of vibration. Sloshing motion and parametric oscillations[3] are not considered in the present treatment.

A basic assumption for the new type of element is that of a uniform internal pressure. Equilibrium of an element with respect to its internal pressure is by forces acting at nodes in the usual fashion of discrete element procedure. The resulting liquid idealization is statically an unstable structure. This reflects a true liquid characteristic, as a distinguishing property of a liquid is its instability as a static structure.

In concept, the new idealization of an incompressible liquid is a simplification in liquid vibration theory. Because an incompressible liquid is considered as just another structure along with plates, solid bodies, shells, skin-stringer shells, etc., it can be represented by the same general discrete element idealization procedure. These elements have common physical characteristics. A mathematical treatment of the theory of discrete elements from a general point of view is presented in Ref. 4.

The liquid element idealization leads to a method for computing a liquid mass matrix with respect to the container coordinates. It is an important result that the liquid coordinates are eliminated. This is done by a simple coordinate transformation. That is to say, the discrete element idealization is not used as the basis for analysis in the usual manner. Instead the mass matrix of the empty vehicle is modified by terms from a reduced liquid mass matrix. The resulting characteristic value problem of the vehicle containing liquid is the same size as for the empty vehicle.

The liquid elements can be easily modified for compressibility, and this is considered in the Appendix. The liquid coordinates are not eliminated in the case of compressibility. The assumption of incompressibility makes the characteristic value problem easier and, furthermore, is justified for liquid-propellant missiles by results from Ref. 2. These results show that calculated missile modes and frequencies based on incompressibility agree well with experimental data for the lower modes of interest.

The new type liquid element is explained by discussing properties of several different shaped elements. It is shown that liquid elements at irregular container boundaries such as in a dome present no special problem.

Formulation of the Liquid Structural Element

First consider a cubical liquid element which is the simplest element and which could be used for liquid in a container of rectangular cross section. The element can be considered as made up of tension members (tension members can take compression). However, the group of tension members are considered as one element. Consider a cube of liquid with three tension members of equal length, AA', BB', and CC' (Fig. 1). The points shown are the nodes where element forces act and where adjacent elements of an array are at-

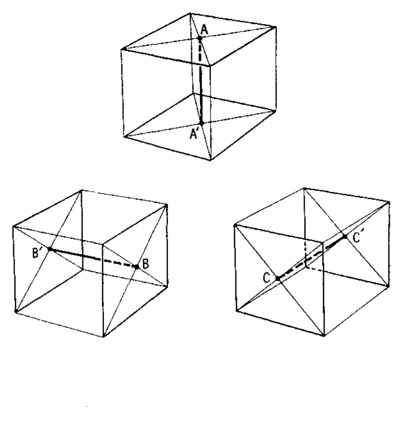

Fig. 1 Cubical element.

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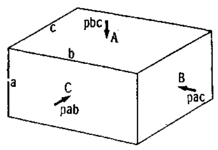

Fig. 2 Rectangular element.

tached. The liquid is incompressible. Therefore, the sum of the deformation of the members is

$$\delta_a + \delta_b + \delta_c = 0$$

The stress (pressure) in a liquid at a point in the three perpendicular directions are equal. Therefore, the forces in the three members must be equal

$$P_a = P_b = P_c$$

These three members are considered as one element with the generalized force P, where

$$P_a = P_b = P_c = P$$

The corresponding generalized displacement is

$$\delta_a + \delta_b + \delta_c$$

It is obvious that the flexibility corresponding to the generalized force P is zero, and that there is no coupling between liquid elements.

If adjacent elements are cubes, the mass matrix can be formed by lumping the mass of liquid included by the cube at nodes A and A' (Fig. 1) and assigning these masses to degrees of freedom in the AA' direction. Similarly, other portions of the mass matrix are formed for degrees of freedom at the remaining nodes. There is just one mass degree of freedom at a node and this is in the direction of a line joining nodes on opposite faces of the cube.

The incompressibility condition provides an equation of constraint for each element and these equations reduce the degrees of freedom in the solution of a liquid structure idealization. This is demonstrated for a simple example that is worked out in a later section.

A liquid element in the form of a rectangular parallelopiped with internal pressure p would be subjected to forces at the nodes as shown in Fig. 2. For this element, the relation between element deformation from the condition of incompressibility gives the following constraint equation:

$$\delta_a/a + \delta_b/b + \delta_c/c = 0$$

The liquid in a cylinder of circular cross section could be idealized by an array of elements attached to each other at nodes as shown in Fig. 3. This array consists of two kinds of elements which are the cylindrical and triangular elements shown in more detail in Fig. 4.

The forces acting on the elements corresponding to an internal pressure p act at the nodes as indicated. The forces for each element are, of course, in equilibrium.

The constraint equations from the condition of incompressibility for the cylindrical and triangular elements are, respectively,

$$(R - r)h \cos\alpha(\delta_b + \delta_c) - 2hR \sin\alpha \cos\alpha\delta_R + 2hr \sin\alpha \cos\alpha\delta_r - [2r(R - r) \sin\alpha \cos\alpha + (R - r)^2 \sin\alpha \cos\alpha]\delta_h = 0$$

and

$$Rh \cos\alpha(\delta_b + \delta_c) - 2hR \sin\alpha \cos\alpha\delta_R - R^2 \sin\alpha \cos\alpha\delta_h = 0$$

The deformations are with reference to the nodes and, disregarding sign, in the direction of the forces at the nodes. Deformations δ_R, δ_r, and δ_h are arbitrarily positive and δ_b and δ_c negative when the corresponding dimension of an element is increased. The work done by the element forces moving through the element deformations is, of course, zero for all elements which have been discussed.

The mass matrix is formed by lumping liquid masses at the nodes. Again there is just one mass degree of freedom at a node.

Instead of the twelve triangular elements (Fig. 3), one element in the shape of a twelve-sided regular polygon could be used. The polygon element due to internal pressure p is subjected to a radial force $2phR \sin\alpha \cos\alpha$ at each of the twelve nodes which are on a circle of radius, R (Fig. 4) and to a longitudinal force $12pR^2 \cos\alpha \sin\alpha$ at a node in the center of each end face. Each of the twelve nodes has one radial degree of freedom and the mass matrix due to the liquid included by the polygon is formed by lumping $\frac{1}{6}$ of this mass at each of the twelve nodes. This satisfies mass degree-of-freedom orthogonality in the radial plane. Additional mass lumping is at a node on each end face corresponding to a longitudinal degree of freedom. The constraint equation is easily found from the condition that the work of the element forces at the nodes during an element deformation is zero for incompressibility.

Liquid elements in a dome section are shown in Fig. 5. The liquid elements in the interior of the dome can be the same as the elements in the cylindrical section. The development of the boundary elements proceeds along the same lines as that for the interior elements.

Some of the forces acting on a typical boundary element because of its internal pressure are shown in Fig. 5. Node locations as before are determined from the condition of element equilibrium due to the internal pressure. Liquid mass

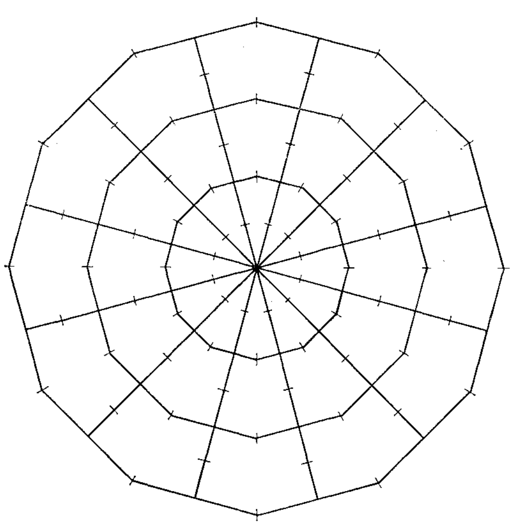

Fig. 3 Cross-sectional array of elements.

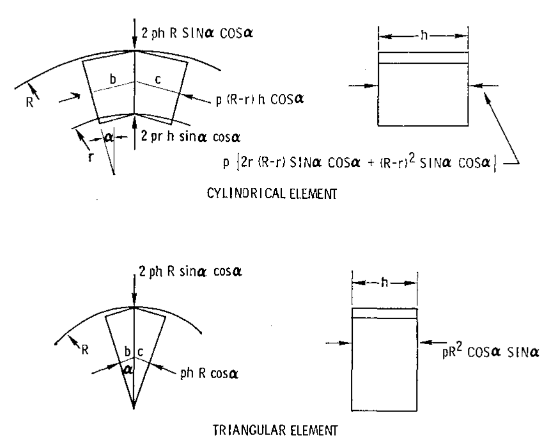

Fig. 4 Cylindrical and triangular elements.

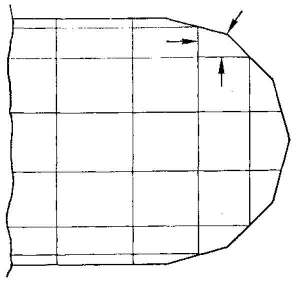

Fig. 5 Elements in a dome section.

is not lumped at a boundary node. Instead, the liquid mass included by the element is lumped at nodes on the vertical and horizontal sides corresponding to degrees of freedom in the direction of the forces acting on these sides. Thus orthogonality of these degrees of freedom is satisfied. There are two more nodes on a typical boundary element, one on each side, common to a like element. An element force and a mass degree of freedom acts at each of these nodes similarly to the nodes on the sides of the elements shown in Fig. 4. Again the constraint equation is found from the condition of zero work by the element forces during an element deformation.

Formulation of the Reduced Liquid Mass Matrix

The liquid discrete element idealization leads to a method for computing a mass matrix for the liquid with respect to the container coordinates. The matrix formulation of this procedure is now derived.

The original liquid mass matrix M corresponds to the total coordinates,

$$\{x\} = \begin{Bmatrix} x_0 \\ \hline x_a \\ \hline x_c \end{Bmatrix}$$

where x_0 are ignorable coordinates,[5] x_a are structure coordinates and x_c are coordinates eliminated by the constraint equations. Coordinates x_0 and x_c are those of the liquid elements. This mass matrix could be diagonal formed by a simple lumping procedure for each liquid element as previously discussed or the matrix could contain off diagonal terms from a consistent mass formulation.[6] In any case, the problem is to derive the liquid mass matrix corresponding to the coordinates, x_a. For this purpose, only the container coordinates need to be included in the x_a matrix.

The constraint equations for the liquid elements can be put in the form

$$[T_1|T_2] \begin{Bmatrix} x_p \\ \hline x_c \end{Bmatrix} = \{0\}$$

where

$$\{x_p\} = \begin{Bmatrix} x_0 \\ \hline x_a \end{Bmatrix}$$

It follows that

$$\{x\} = \begin{bmatrix} [I] \\ \hline [-T_2^{-1}T_1] \end{bmatrix} \{x_p\} = A\ \{x_p\}$$

The liquid mass matrix corresponding to the coordinates x_p is A^tMA. This mass matrix can be written

$$\begin{bmatrix} M_1 & M_2 \\ M_3 & M_4 \end{bmatrix}$$

as defined by the equations of motion

$$\begin{bmatrix} M_1 & M_2 \\ M_3 & M_4 \end{bmatrix} \begin{Bmatrix} \ddot{x}_0 \\ \hline \ddot{x}_a \end{Bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & K \end{bmatrix} \begin{Bmatrix} x_0 \\ \hline x_a \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

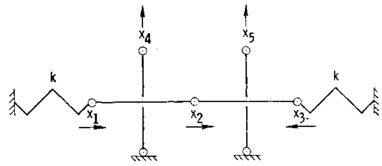

Fig. 6 Container and liquid idealization.

It follows that

$$M_1\{x_0\} + M_2\{x_a\} = \{0\}$$

These are dynamic constraint equations. Because the liquid idealization is statically unstable, equilibrium of certain nodes during vibration can only be satisfied by inertia forces. This puts a constraint on the coordinates as represented by the above system of equations for all nonzero frequency modes. From these equations, it follows that

$$\{x_p\} = \begin{bmatrix} [-M_1^{-1}M_2] \\ \hline [I] \end{bmatrix} \{x_a\} = B\{x_a\}$$

The liquid mass matrix corresponding to the coordinates x_a is B^tA^tMAB which is what we set out to find.

Mode Shapes and Frequencies of a Simple Container and Liquid Structure

Mode shapes and frequencies are found for the simple structural idealization shown in Fig. 6. The idealization consists of two springs and two planar liquid elements. The springs support the liquid elements laterally, representing the action of flexible container walls. The liquid elements represent a liquid subjected to planar motion and are otherwise the same as the three dimensional cubical elements previously discussed.

At each of the five coordinates it is assumed the mass is m. Using for each element the constraint equation from the condition of incompressibility, the equations of equilibrium are

$$2m\ddot{x}_1 - m\ddot{x}_2 + kx_1 = 0,\ -m\ddot{x}_1 + 3m\ddot{x}_2 + m\ddot{x}_3 = 0$$
$$m\ddot{x}_2 + 2m\ddot{x}_3 + kx_3 = 0$$

Eliminating an ignorable coordinate, the problem is reduced to

$$5m\ddot{x}_1 + m\ddot{x}_3 + 3kx_1 = 0,\ m\ddot{x}_1 + 5m\ddot{x}_3 + 3kx_3 = 0$$

The two frequencies from these equations are $k/2m$ and $3k/4m$. The resulting mode shapes, including the zero frequency mode, are shown in Fig. 7.

This simple idealization illustrates all the basic properties of a container and liquid idealization that in practice could

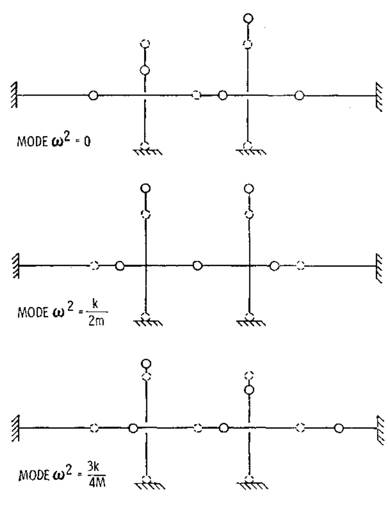

Fig. 7 Mode shapes of the container and liquid idealization.

consist of hundreds of elements. The idealization is statically unstable and can only support certain sets of external forces. For example, vertical forces at nodes 4 and 5, (Fig. 6) must be equal in magnitude and direction. This corresponds to the static force condition of a real liquid in a container at its top surface where the pressure must be uniform.

An additional property is that the idealization has a zero frequency mode. The idealization has coordinates which are eliminated by constraint equations and has an ignorable coordinate. The final liquid mass matrix is reduced to one with respect to only the container coordinates.

Concluding Remarks

A new structural discrete element idealization of an incompressible liquid is introduced which has the following real liquid properties.

1) The stress (pressure) in a cubical liquid element as an example, are equal in the three perpendicular directions.

2) Considering the weight of the liquid as lumped at nodes, the resulting hydrostatic pressure at a constant height in a container is equal in all liquid elements of the same height.

3) The liquid idealization does not add additional stiffening to the container.

4) The liquid idealization is statically unstable and can only support certain sets of static external forces.

5) The displacements of the liquid idealization which results from a stable system of static external forces depend completely on the container structure displacements.

6) Some of the vibration modes of the container and liquid idealization are zero frequency modes, the number depending on the number of liquid elements. This reflects a true phenomenon, as a liquid has an infinite number of zero frequency modes.

Appendix: Compressible Fluid Elements

A compressible rectangular parallelopiped fluid element (Fig. 2) will be considered. The relation between vibratory element pressure p and element unit volume change, $\Delta v/v$, is

$$p = E\Delta v/v$$

where E is the fluid bulk modulus. It follows that the relation between p and element grid point displacements is,

$$p = E(\delta_a/a + \delta_b/b + \delta_c/c)$$

Similar expressions can be written for fluid elements of other shapes.

Fluid mass is lumped at grid points in the same manner as for the incompressible elements. Again, each grid point has only one mass degree of freedom.

Dynamic solution of a structural idealization of compressible fluid elements can be based on a method using a system of recursive equations. Such a method applicable to the compressible idealization is fully explained and proven to be satisfactory in Ref. 7. There a stress wave problem in a continuum material was solved by discrete element idealization and recursive equations. Results obtained compare very well with those from an analytic solution.

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